Lecture 2: Basic mathematical concept

Brief introduction on:

1. Topological surface
2. Riemannian surface

Topological surface

- interested in the topology / genus of the surface ONLY.
- not equipped with a metric measuring distance

Riemannian surface

- Equipped with a metric measuring distance
- Topology / genus can be discovered by Gauss-Bonnet Theorem.

$$\int_M K \, dA = 2\pi \chi(M), \quad \chi(M) = 2-2g$$

K: Gaussian curvature; $\chi(M)$: Euler char.

Topological surfaces:

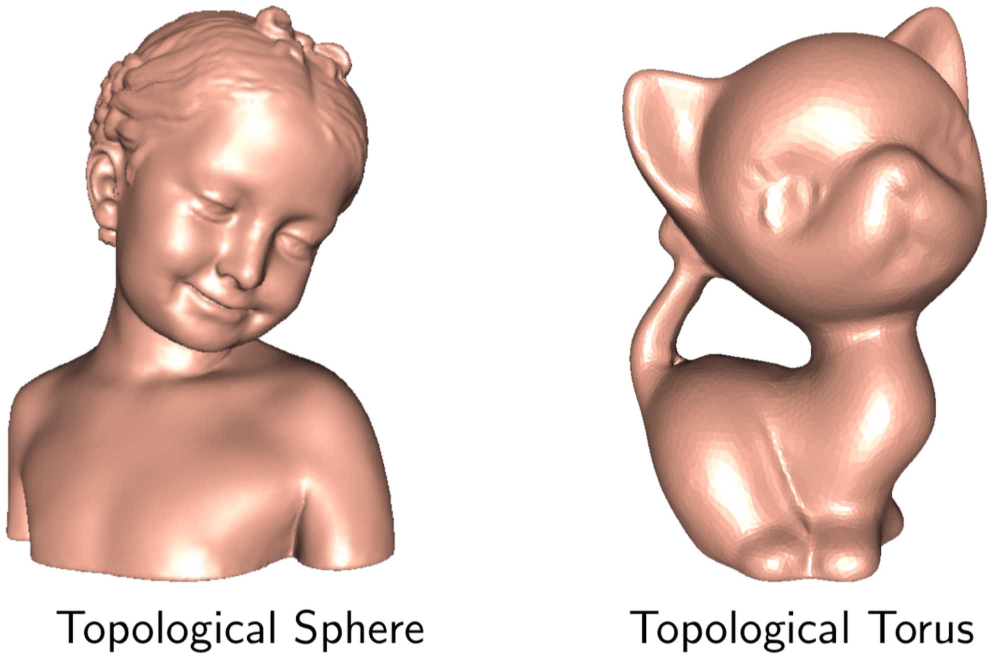


Figure: How to differentiate the above two surfaces.

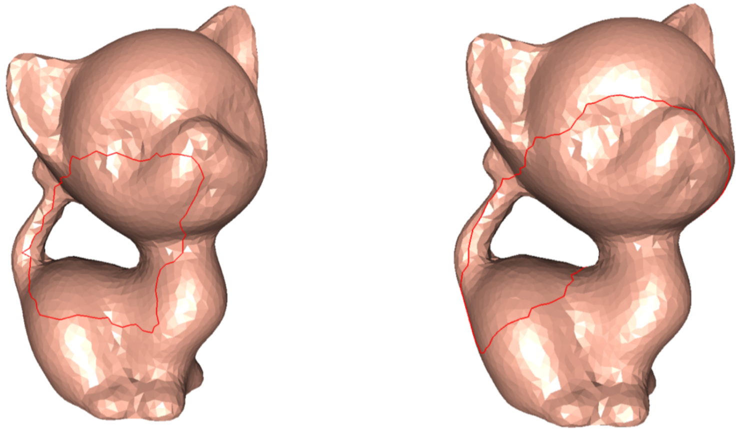


Figure: Check whether all loops on the surface can shrink to a point.

All oriented compact surfaces can be classified by their genus g and number of boundaries b. Therefore, we use (g, b) to represent the topological type of an oriented surface S.

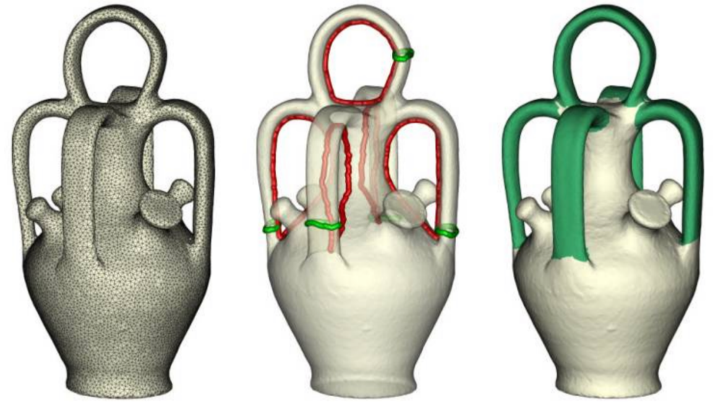


Figure: Handle detection by finding the handle loops and the tunnel loops.

Remark: Topological surface S can be determined by the first homotopy group.

Suppose $q \in S$ is a base point, all oriented loop can be classified by homotopy and hence form a homotopic class.

All homotopic classes form the fundamental group / first homotopic class of S. Denote it by $\pi_1(S, q)$.

Definition: Let $\gamma_1, \gamma_2 : [0,1] \to S$ be two curves. A homotopy connecting γ_1 and γ_2 is a continuous mapping $F : [0,1] \times [0,1] \to S$, such that:

$$F(0,t) = \gamma_1(t) \quad \text{and} \quad F(1,t) = \gamma_2(t).$$

γ_1 is said to be homotopic to γ_2 if there exists a homotopy between them.

Definition: A closed curve (loop) through p is a curve such that $\gamma(0) = \gamma(1) = p$.

Lemma: Homotopy relation is an equivalence relation.

Remark: The homotopy class of a loop γ is denoted by $[\gamma]$.

(If $\gamma_1 \in [\gamma]$, then: γ_1 is homotopic to γ)

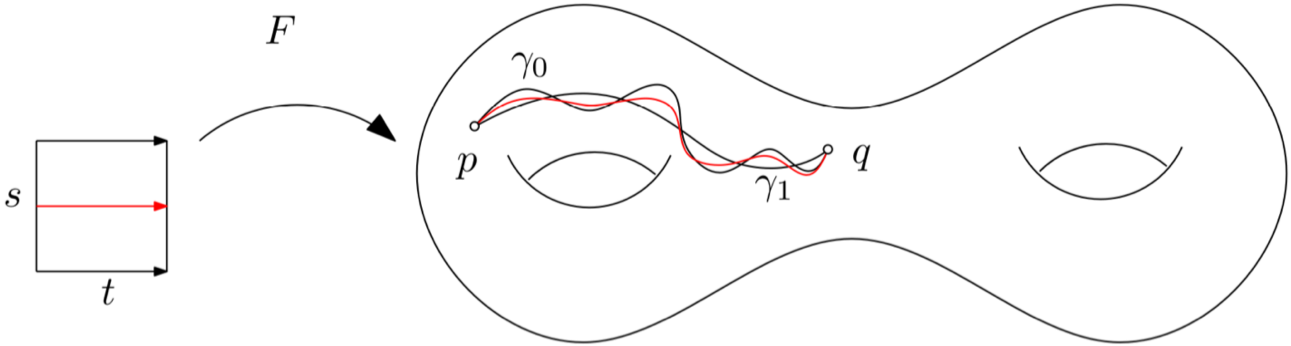


Figure: Path homotopy.

Definition: Let γ_0, γ_1 be two loops through p. The product of two loops is defined as:

$$\gamma_0 \cdot \gamma_1 (t) = \begin{cases} \gamma_0(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_1(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

The loop inverse is defined as:

$$\gamma^{-1}(t) = \gamma(1-t)$$

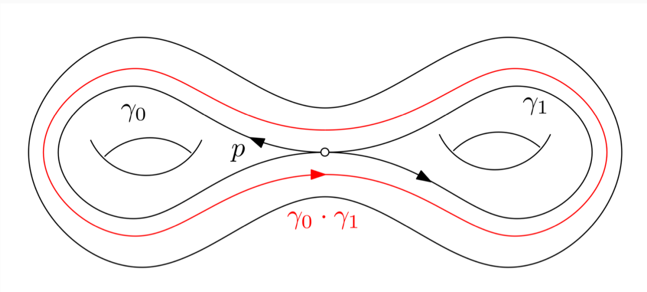


Figure: Loop product

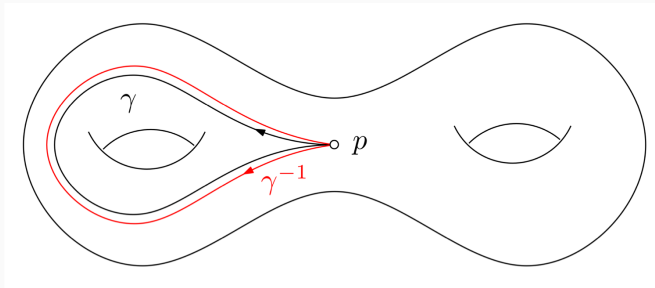
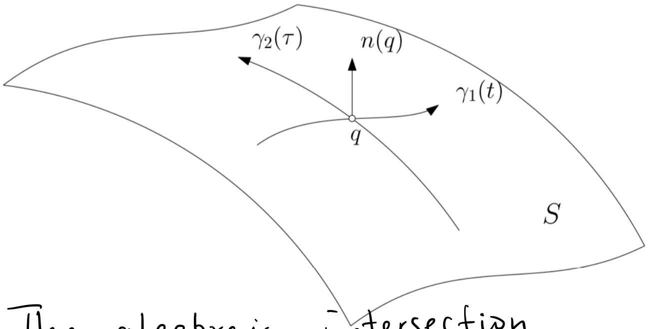


Figure: Loop inversion

Definition: (Intersection index)



Suppose γ_1 and γ_2 intersects at q. That's, $\gamma_1(t) = \gamma_2(\tau) = q$.

Then: the intersection index at q is:

$$\mathrm{Ind}(\gamma_1, \gamma_2, q) \stackrel{def}{=} \begin{cases} 1 & \text{if } \frac{d\gamma_1}{dt} \times \frac{d\gamma_2}{dt} \cdot \vec{n} > 0 \\ -1 & \text{if } \frac{d\gamma_1}{dt} \times \frac{d\gamma_2}{dt} \cdot \vec{n} < 0 \\ 0 & \text{if } \frac{d\gamma_1}{dt} \times \frac{d\gamma_2}{dt} \cdot \vec{n} = 0 \end{cases}$$

The algebraic intersection number of γ_1 and γ_2 is defined as:

$$\gamma_1 \cdot \gamma_2 \stackrel{def}{:=} \sum_{q_i \in \gamma_1 \cap \gamma_2} \mathrm{Ind}(\gamma_1, \gamma_2, q_i)$$

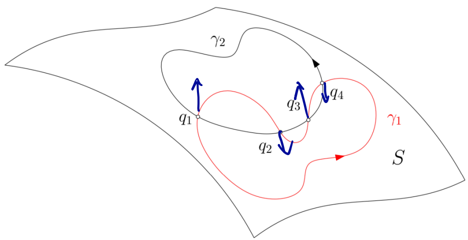


Figure: Algebraic intersection number

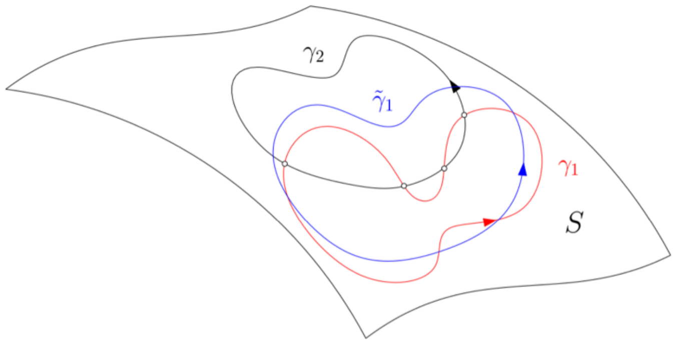


Figure: Algebraic intersection number

Algebraic Intersection Number Homotopy Invariance

Suppose γ_1 is homotopic to $\tilde{\gamma}_1$, then the algebraic intersection number

$$\gamma_1 \cdot \gamma_2 = \tilde{\gamma}_1 \cdot \gamma_2.$$

Proof: Exercise

Definition (Canonical Basis)

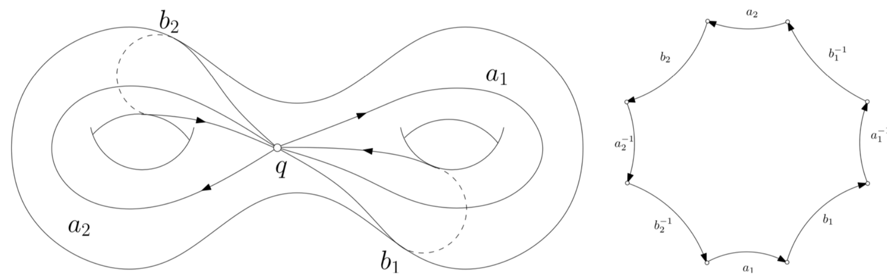
Suppose S is a compact, oriented surface, there exists a set of generators of the fundamental group $\pi_1(S, p)$,

$$G = \{[a_1], [b_1], [a_2], [b_2], \cdots, [a_g], [b_g]\}$$

such that

$$a_i \cdot b_j = \delta_i^j, a_i \cdot a_j = 0, b_i \cdot b_j = 0,$$

where $a_i \cdot b_j$ represents the algebraic intersection number of loops a_i and b_j, δ_{ij} is the Kronecker symbol, then G is called a set of canonical basis of $\pi_1(S, p)$.



Remark: We'll learn how to find a_i's, b_j's ∋ we can cut and flatten!

Universal covering space

Definition (Covering Space) Let S and $\tilde{S}$ be topological spaces. A continuous map $p: \tilde{S} \to S$ is a <u>covering map</u> if:

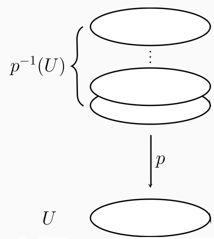
(1) For each $q \in S$, $\exists$ neighbourhood $\mathcal{U}$ of q such that

$$p^{-1}(\mathcal{U}) = \dot{\bigcup_i} \tilde{\mathcal{U}}_i$$ is a disjoint union of open sets $\tilde{\mathcal{U}}_i$

(2) $p|_{\tilde{\mathcal{U}}_i} : \tilde{\mathcal{U}}_i \to \mathcal{U}_i$ is a homeomorphism. for $\forall i$.

Then: $\tilde{S}$ is called a <u>covering space</u>.

If $\tilde{S}$ is simply-connected, then $\tilde{S}$ is called a <u>universal covering space</u>.



Definition: (Deck Transformation)

The automorphism of $\tilde{S}$, $\tau: \tilde{S} \to \tilde{S}$, is called a deck transformation if they satisfy $p \circ \tau = p$.

All deck transformations form a group, the covering group, and denoted as $Deck(\tilde{S})$.

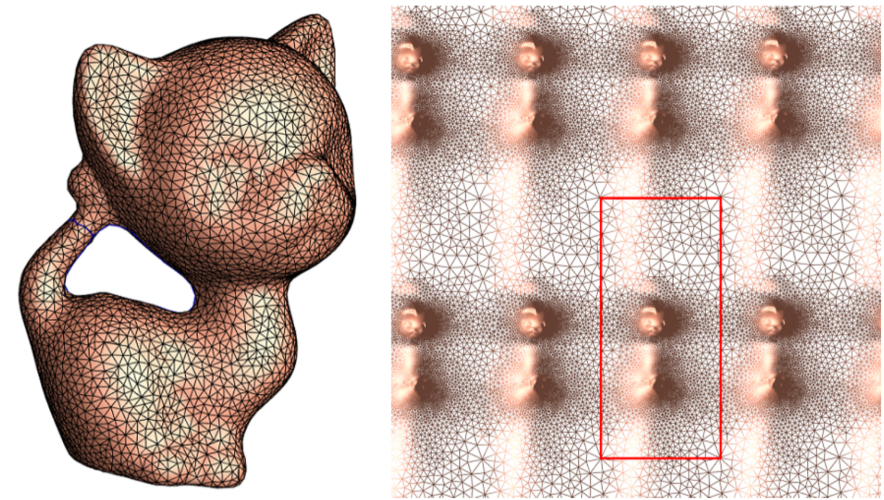
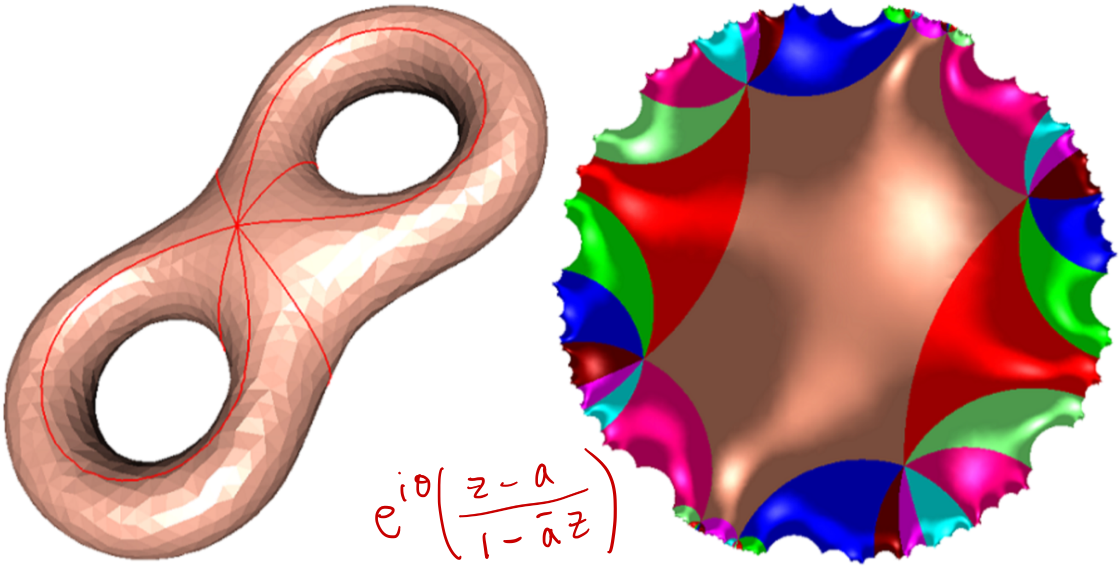


Figure: Universal Covering Space

$Deck(\tilde{S})$

"

Space of translations from one fundamental domain to another.



$$e^{i\theta}\left(\frac{z-a}{1-\bar{a}z}\right)$$

Figure: Universal Covering Space of a genus two surface.

Deck$(\tilde{S})$ = Space of Möbius transformations.

Smooth manifold

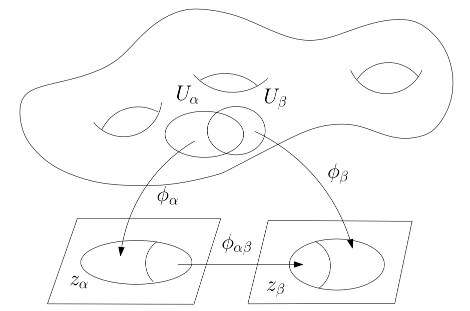
Definition (Manifold)

A manifold is a topological space M covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a coordinate chart of M. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of M. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition maps $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ are smooth, then the manifold is a differential manifold or a smooth manifold.



Definition (Tangent Vector)

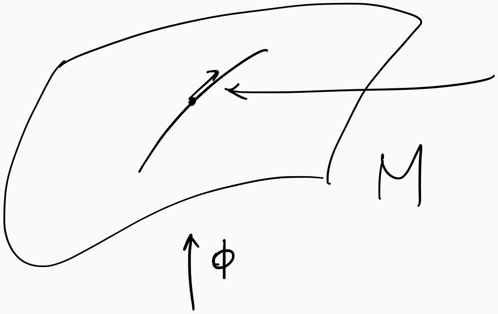
A tangent vector ξ at the point p is an association to every coordinate chart $(x^1, x^2, \cdots, x^n)$ at p an n-tuple $(\xi^1, \xi^2, \cdots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \cdots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p)\xi^j.$$

A smooth vector field ξ assigns a tangent vector for each point of M, it has local representation

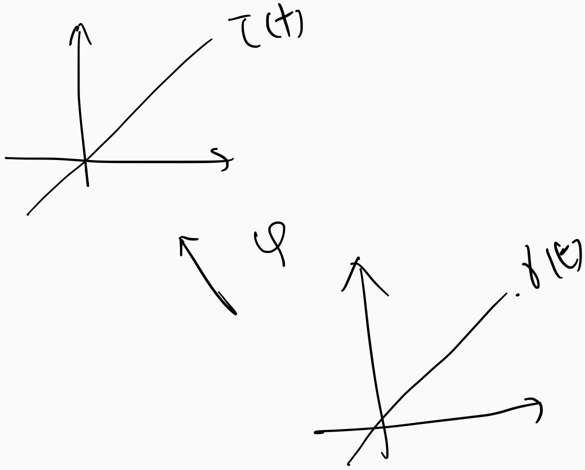
$$\xi(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \xi_i(x^1, x^2, \cdots, x^n)\frac{\partial}{\partial x_i}.$$

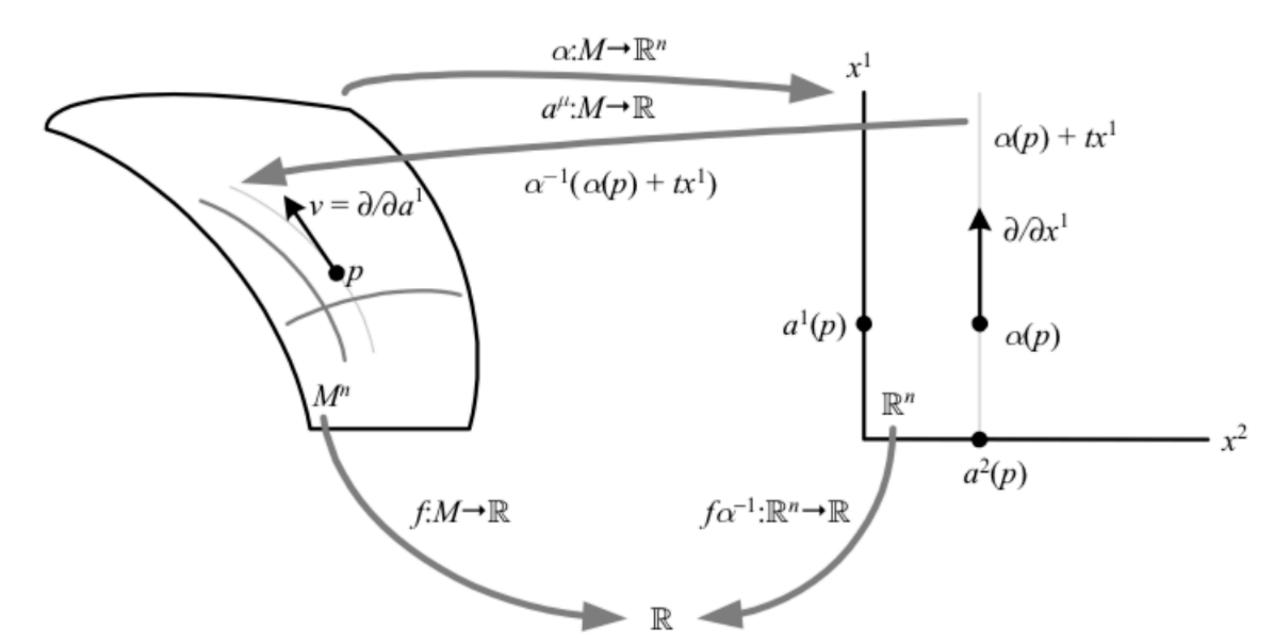
$\{\frac{\partial}{\partial x_i}\}$ represents the vector fields of the velocities of iso-parametric curves on M. They form a basis of all vector fields.



$$\frac{d}{dt}\Big|_{t=0} \phi(\tau(t))$$

$$= \frac{d}{dt}\Big|_{t=0} \phi(\tau \circ \gamma(t))$$




$\alpha: M \to \mathbb{R}^n$
$a^{\mu}: M \to \mathbb{R}$
x^1
$\alpha(p) + tx^1$
$\alpha^{-1}(\alpha(p) + tx^1)$
$v = \partial/\partial a^1$
$\partial/\partial x^1$
p
$a^1(p)$
$\alpha(p)$
M^n
$\mathbb{R}^n$
x^2
$a^2(p)$
$f: M \to \mathbb{R}$
$f\alpha^{-1}: \mathbb{R}^n \to \mathbb{R}$
$\mathbb{R}$